On realization of Boolean functions by repetition-free formulas

I. K. Sharankhaev

Buriyat State University, Russia

Abstract

We study representation of Boolean functions by formulas. We proved necessary and sufficient condition of representation of Boolean functions by repetition-free formulas in the base $\{\vee, \cdot, -, 0, 1, x_1(x_2 \vee x_3) \vee x_3x_4\}$.

Keywords: Boolean function, formula, base, repetition-free function, weak-repetition function, almost elementary base.

1 Introduction

In this article we study realization of Boolean functions by repetition-free formulas in the finite full set (base).

We preface the description of main results with the needed definitions and notation. The definitions of all notions which are not given here can be found, for example, in [1]. We use the following notation: variables are denoted by the symbols x, y, z, u, v, maybe with subscripts; constants are denoted by the symbols σ, $\sigma_1, \dots \sigma_n$; the symbol $\tilde{x}$ denotes the tuple $(x_1, \dots, x_n)$; $|\tilde{x}|$ is the length of a tuple $\tilde{x}$; rank f is the rank of a function f; $\rho(f)$ denotes the set of all essential variables of a function f; $\delta(f)$ denotes the set of all fictitious variables of a function f; P_B denotes the set of all repetition-free functions over a base B; S_B denotes the set of all weak-repetition functions over a base B;

$$x^\sigma = \begin{cases} x, \text{ if } \sigma = 1; \\ \bar{x}, \text{ if } \sigma = 0. \end{cases}$$

A formula Φ over a base B is called *repetition-free* if each variable occurs in Φ at most once.

A Boolean function f is called *repetition-free* in the base B whenever there exists a repetition-free formula Φ over B representing f. Otherwise, f has *repetitions* in B.

The function obtained from $f(x_1, \ldots, x_n)$ by the substitution of a constant σ for a variable x_i is called the *residual* function and is denoted by $f^\sigma_{x_i}$. This definition is extended to a subset of variables by induction.

A variable x_i of a function f is called *fictitious* if $f^0_{x_i} = f^1_{x_i}$ and *essential* otherwise.

The *rank* of a function f is the number of essential variables of f. By the rank of a base is meant the maximum rank of the functions in the base.

We call

$$B_0 = \{\vee, \cdot, -, 0, 1\}$$

the *elementary* base, and every $B_0 \cup \{f\}$, where f is a weak-repetition function in B_0, an *almost elementary* base.

In the article [2] Stetsenko found the almost elementary bases. We introduce notations for almost elementary bases:

$$B_{1,n} = B_0 \cup \{g_{1,n}\}, \text{ where } g_{1,n} = x_1 \cdot \ldots \cdot x_n \vee \bar{x}_1 \cdot \ldots \cdot \bar{x}_n, n \geq 2,$$

$$B_{2,n} = B_0 \cup \{g_{2,n}\}, \text{ where } g_{2,n} = x_1(x_2 \vee \ldots \vee x_n) \vee x_2 \cdot \ldots \cdot x_n, n \geq 3,$$

$$B_{3,n} = B_0 \cup \{g_{3,n}\}, \text{ where } g_{3,n} = x_1(x_2 \vee x_3 \cdot \ldots \cdot x_n) \vee x_2 \cdot \bar{x}_3 \cdot \ldots \cdot \bar{x}_n, n \geq 3,$$

$$B_4 = B_0 \cup \{g_4\}, \text{ where } g_4 = x_1(x_2 \vee x_3) \vee x_3 x_4,$$

$$B_5 = B_0 \cup \{g_5\}, \text{ where } g_5 = x_1(x_2 \vee x_3 x_4) \vee x_5(x_3 \vee x_2 x_4).$$

The articles [3 – 11] yield necessary and sufficient conditions for the repetition freeness of Boolean functions in B_0, $B_{1,2}$, $B_{3,3}$, B_5, $B_{1,n}$, where n is odd, $B_{2,n}$, where $n \geq 3$. Here we offer a criterion for the repetition freeness of Boolean functions in B_4.

Say that two functions f and g are related by $\preceq$, and write $f \preceq g$, whenever $f(\tilde{\sigma}) \leq g(\tilde{\sigma})$ for every tuple $\tilde{\sigma}$. A function f is called *generalized monotone* in x whenever either $f_x^0 \preceq f_x^1$ or $f_x^0 \succeq f_x^1$. If f is a generalized monotone function in x then we put $f \in M_x$ for brevity.

Two functions f and g are of the same generalized type whenever

$$f(x_1, \ldots, x_n) = g^{\sigma}(x_{i_1}^{\sigma_1}, \ldots, x_{i_n}^{\sigma_n}),$$

where $(i_1, \ldots, i_n)$ is some permutation of the integers from 1 to n. It is obvious that the relation of being of the same generalized type is an equivalence on the set of all Boolean functions.

The derivative of a function $f(x_1, \ldots, x_n)$ with respect to x_i is the function

$$f'_{x_i} = f_{x_i}^0 \oplus f_{x_i}^1.$$

The concept of the derivative of a function with respect to a variable extends inductively to the sets of variables as follows:

$$\frac{\partial f}{\partial x_{i_1} \ldots \partial x_{i_s}} = \frac{\partial \left(\dfrac{\partial f}{\partial x_{i_1} \ldots \partial x_{i_{s-1}}} \right)}{\partial x_{i_s}}.$$

A function is called *odd* if the number of tuples at which it assumes 1 is odd, and even otherwise.

A set P of Boolean functions that contains the identity function is called *hereditary* whenever given $f \in P$ every residual function f_x^σ belongs to P.

A set P of Boolean functions is called *invariant* whenever given two functions $f(\tilde{u}, y)$, $g(\tilde{v}) \in P$ with $\tilde{u} \cap \tilde{v} = \emptyset$, we have $f(\tilde{u}, g(\tilde{v})) \in P$.

2 Auxiliary Statements

In proving the main result we will use only the following statements:

Proposition 1 ([5]). *A set P of Boolean functions is hereditary and invariant if and only if P is the set of all repetition-free functions over some base B.*

Corollary 1. *Given a hereditary invariant set P of Boolean functions and a base B with $B \subseteq P$ and $S_B \cap P = \emptyset$, it follows that $P_B = P$.*

Therefore, in order to prove that some set P of Boolean functions coincides with the set of all repetition-free functions over some base B it suffices to show that P enjoys the properties of heredity and invariance, and verify that all weak-repetition functions in B do not belong to P.

Proposition 2 ([12]). *The following system of Boolean functions is a complete system of representatives of the equivalence classes with respect to the relation of being of the same generalized type for weak-repetition Boolean functions in the almost ele-*

mentary base B_4:

$$x_1(x_2 \vee x_3x_4) \vee x_5(x_3 \vee x_2x_4);$$
$$x_1(x_2 \vee \ldots \vee x_n) \vee x_2 \cdot \ldots \cdot x_n,\ n \geq 3;$$
$$x_1(x_2 \vee x_3 \cdot \ldots \cdot x_n) \vee x_2 \cdot \bar{x}_3 \cdot \ldots \cdot \bar{x}_n,\ n \geq 3;$$
$$x_1 \cdot \ldots \cdot x_n \vee \bar{x}_1 \cdot \ldots \cdot \bar{x}_n,\ n \geq 2;$$
$$\bar{x}_1 g(x_2, \ldots, x_5) \vee\ x_1 g(x_3, x_2, x_5, x_4);$$
$$\bar{x}_1 g(x_2, \ldots, x_5) \vee\ x_1x_3x_5(x_2 \vee x_4);$$
$$\bar{x}_1 g(x_2, \ldots, x_5) \vee\ x_1(x_2x_3 \vee x_4x_5);$$
$$\bar{x}_1 g(x_2, \ldots, x_5) \vee\ x_1x_3(x_2 \vee x_4x_5);$$
$$\bar{x}_1x_2 g(x_3, \ldots, x_6) \vee\ x_1 g(x_2x_3, x_4, x_5, x_6).$$

3 The Main Theorem

A function f will be called 4-$soft$ whenever either rank f < 2, or for every $x \in \rho(f)$ we have $f \in M_x$ and one of the following conditions holds:

1. for some constant σ conditions $\delta(f) \subsetneq \delta(f_x^\sigma)$ and $\delta(f) = \delta(f_x^{\bar{\sigma}})$ are true, and if $\delta(f_x^\sigma) \setminus \delta(f) = \{y\}$, then $\delta(f) = \delta(f_y^0) = \delta(f_y^1)$ is not true,

2. $\delta(f) \subsetneq \delta(f_x^0)$, $\delta(f) \subsetneq \delta(f_x^1)$ and there exists $y \in \rho(f'_x)$ such that $\delta(f'_x) \subsetneq \delta((f'_x)'_y)$,

3. $\delta(f) \subsetneq \delta(f'_x)$.

A function f will be called *hereditarily* 4-$soft$ if f and all its residual functions are 4-soft.

The following theorem yields a repetition-free criterion.

Theorem 1. *A function f is repetition-free in the base B_4 if and only if f is hereditarily 4-soft.*

Proof. In order to prove the theorem we use the method that is based on Proposition 1. Denote by P the set of all hereditarily 4-soft functions. The set P is hereditary by definition, and so let us demonstrate the invariance of P.

Suppose that $f(\tilde{u},\tilde{v}) = g(\tilde{u},h(\tilde{v}))$ with $g(\tilde{u},y)$, $h(\tilde{v}) \in P$. If $\tilde{u} = \emptyset$ or $|\tilde{v}| = 1$ then f is of the same generalized type as g or h; thus, f is hereditarily 4-soft. Assume henceforth that $\tilde{u} \neq \emptyset$ and $|\tilde{v}| > 1$.

1. Let $x \in \tilde{v}$. If for some constant σ conditions $\delta(h) \subsetneq \delta(h_x^\sigma)$ and $\delta(h) = \delta(h_x^{\bar{\sigma}})$ are true, then $\delta(f) \subsetneq \delta(f_x^\sigma)$ and $\delta(f) = \delta(f_x^{\bar{\sigma}})$ holds. And if $\delta(f_x^\sigma) \setminus \delta(f) = \{y\}$, then $\delta(h_x^\sigma) \setminus \delta(h) = \{y\}$, and $\delta(f) = \delta(f_y^0) = \delta(f_y^1)$ is not true, since $\delta(h) = \delta(h_y^0) = \delta(h_y^1)$ is not true.

   Let $\delta(h) \subsetneq \delta(h_x^0)$ and $\delta(h) \subsetneq \delta(h_x^1)$. There exists a variable z such that $\delta(h'_x) \subsetneq \delta((h'_x)'_z)$, therefore $\delta(f'_x) \subsetneq \delta((f'_x)'_z)$.

   Let $\delta(h) \subsetneq \delta(h'_x)$. Since $f'_x = g'_y(\tilde{u},y) \cdot h'_x(\tilde{v})$ is true, the strict inclusion $\delta(f) \subsetneq \delta(f'_x)$ holds.

2. Let $x \in \tilde{u}$. If for some constant σ conditions $\delta(g) \subsetneq \delta(g_x^\sigma)$ and $\delta(g) = \delta(g_x^{\bar{\sigma}})$ are true, then $\delta(f) \subsetneq \delta(f_x^\sigma)$ and $\delta(f) = \delta(f_x^{\bar{\sigma}})$ holds. And if $\delta(g_x^\sigma) \setminus \delta(g) = \{y\}$ and $\delta(f_x^\sigma) \setminus \delta(f) = \{y\}$, then $\delta(f) = \delta(f_y^0) = \delta(f_y^1)$ is not true, since $\delta(g) = \delta(g_y^0) = \delta(g_y^1)$ is not true.

Let $\delta(g) \subsetneq \delta(g_x^0)$ and $\delta(g) \subsetneq \delta(g_x^1)$. If there exists a variable z, not equal to y and such that $\delta(g'_x) \subsetneq \delta((g'_x)'_z)$, then $\delta(f'_x) \subsetneq \delta((f'_x)'_z)$. Otherwise we choose arbitrarily a variable $z_1 \in \tilde{v}$ and consider $(f'_x)'_{z_1} = (g'_x)'_y(\tilde{u},y) \cdot h'_{z_1}(\tilde{v})$. Since $\delta(g'_x) \subsetneq \delta((g'_x)'_y)$ is true, then $\delta(g'_x) \subsetneq \delta((g'_x)'_{z_1})$ holds.

Let $\delta(g) \subsetneq \delta(g'_x)$. It is easily seen that $\delta(f) \subsetneq \delta(f'_x)$.

In order to prove that $f \in M_x$ for every variable x, we argue by contradiction. Suppose that $f \notin M_x$; i. e., $f_x^0 \npreceq f_x^1$ and $f_x^0 \nsucceq f_x^1$.

Let $x \in \tilde{u}$. Then there are tuples of constants $\tilde{\sigma}_1$, $\tilde{\sigma}_2$, $\tilde{\sigma}_3$, $\tilde{\tau}_1$, $\tilde{\tau}_2$, $\tilde{\tau}_3$ such that $|\tilde{\sigma}_i| = |\tilde{\tau}_i|$ for every i and

$$g(\tilde{\sigma}_1, 0, \tilde{\sigma}_2, h(\tilde{\sigma}_3)) < g(\tilde{\sigma}_1, 1, \tilde{\sigma}_2, h(\tilde{\sigma}_3)),$$

$$g(\tilde{\tau}_1, 0, \tilde{\tau}_2, h(\tilde{\tau}_3)) > g(\tilde{\tau}_1, 1, \tilde{\tau}_2, h(\tilde{\tau}_3)).$$

Let $h(\tilde{\sigma}_3) = \gamma$, $h(\tilde{\tau}_3) = \delta$, then

$$g(\tilde{\sigma}_1, 0, \tilde{\sigma}_2, \gamma) < g(\tilde{\sigma}_1, 1, \tilde{\sigma}_2, \gamma),$$

$$g(\tilde{\tau}_1, 0, \tilde{\tau}_2, \delta) > g(\tilde{\tau}_1, 1, \tilde{\tau}_2, \delta).$$

Thus, $g \notin M_x$, and we obtain a contradiction.

Let $x \in \tilde{v}$. Similarly, there are tuples of constants $\tilde{\sigma}_1$, $\tilde{\sigma}_2$, $\tilde{\sigma}_3$, $\tilde{\tau}_1$, $\tilde{\tau}_2$, $\tilde{\tau}_3$ such that $|\tilde{\sigma}_i| = |\tilde{\tau}_i|$ for every i and

$$g(\tilde{\sigma}_1, h(\tilde{\sigma}_2, 0, \tilde{\sigma}_3)) < g(\tilde{\sigma}_1, h(\tilde{\sigma}_2, 1, \tilde{\sigma}_3)),$$

$$g(\tilde{\tau}_1, h(\tilde{\tau}_2, 0, \tilde{\tau}_3)) > g(\tilde{\tau}_1, h(\tilde{\tau}_2, 1, \tilde{\tau}_3)).$$

Since $h \in M_x$, it follows that either

$$g(\tilde{\sigma}_1, 0) < g(\tilde{\sigma}_1, 1), \ g(\tilde{\tau}_1, 0) > g(\tilde{\tau}_1, 1),$$

or

$$g(\tilde{\sigma}_1, 1) < g(\tilde{\sigma}_1, 0), \ g(\tilde{\tau}_1, 1) > g(\tilde{\tau}_1, 0).$$

In both cases, we obtain a contradiction with the inclusion $g \in M_x$. Thus, we proved that P is invariant.

Now for the hereditary invariant set P we can find a generating base. It is obvious that $B_4 \subseteq P$. Verify that all weak-repetition functions in B_4 do not belong to P. It suffices to restrict the argument to the functions in Proposition 2 since if the 4-softness property fails to hold for some function then it fails for all functions of the same generalized type either.

(a) Let $f = x_1(x_2 \vee x_3x_4) \vee x_5(x_3 \vee x_2x_4)$. Then

$$f^0_{x_4} = x_1x_2 \vee x_3x_5, f^1_{x_4} = (x_1 \vee x_5)(x_2 \vee x_3), f'_{x_4} = x_1\bar{x}_2x_3\bar{x}_5 \vee \bar{x}_1x_2\bar{x}_3x_5.$$

The functions $f^0_{x_4}$, $f^1_{x_4}$, f'_{x_4} are essential, thus $f \notin P$.

(b) Let $f = x_1(x_2 \vee \ldots \vee x_n) \vee x_2 \cdot \ldots \cdot x_n$, where $n \geq 3$. Then

$$f^0_{x_1} = x_2 \cdot \ldots \cdot x_n, f^1_{x_1} = x_2 \vee \ldots \vee x_n, f'_{x_1} = \overline{x_2 \cdot \ldots \cdot x_n \vee \bar{x}_2 \cdot \ldots \cdot \bar{x}_n}.$$

The functions $f^0_{x_1}$, $f^1_{x_1}$, f'_{x_1} are essential, thus $f \notin P$.

(c) Let $f = x_1(x_2 \vee x_3 \cdot \ldots \cdot x_n) \vee x_2\bar{x}_3 \cdot \ldots \cdot \bar{x}_n$, where $n \geq 3$. The function $f \notin P$, since $f \notin M_{x_3}$.

(d) Let $f = x_1 \cdot \ldots \cdot x_n \vee \bar{x}_1 \cdot \ldots \cdot \bar{x}_n$, where $n \geq 2$. In the same way as in case (c), the function $f \notin P$, since $f \notin M_{x_1}$.

(e) Let $f = \bar{x}_1g(x_2, \ldots, x_5) \vee x_1g(x_3, x_2, x_5, x_4)$. Then

$$f^0_{x_1} = g(x_2, \ldots, x_5), f^1_{x_1} = g(x_3, x_2, x_5, x_4), f'_{x_1} = x_2\bar{x}_3x_4\bar{x}_5 \oplus \bar{x}_2x_3\bar{x}_4x_5.$$

The situation is similar to case (b).

(f) Let $f = \bar{x}_1g(x_2, \ldots, x_5) \vee x_1x_3x_5(x_2 \vee x_4)$. Then

$$f^0_{x_1} = g(x_2, \ldots, x_5), f^1_{x_1} = x_3x_5(x_2 \vee x_4),$$

$$f'_{x_1} = x_2x_3x_4\bar{x}_5 \vee x_2x_3\bar{x}_4\bar{x}_5 \vee x_2\bar{x}_3x_4x_5 \vee x_2\bar{x}_3x_4\bar{x}_5 \vee \bar{x}_2\bar{x}_3x_4x_5.$$

The function f'_{x_1} is odd, that is, it is essential. The situation is similar to case (b).

(g) Let $f = \bar{x}_1 g(x_2, \dots, x_5) \vee \; x_1(x_2x_3 \vee x_4x_5)$. Then

$$f^0_{x_1} = g(x_2, \dots, x_5), f^1_{x_1} = x_2x_3 \vee x_4x_5, f'_{x_1} = x_2\bar{x}_3x_4\bar{x}_5.$$

The situation is similar to case (b).

(h) Let $f = \bar{x}_1 g(x_2, \dots, x_5) \vee \; x_1x_3(x_2 \vee x_4x_5)$. Then

$$f^0_{x_1} = g(x_2, \dots, x_5), f^1_{x_1} = x_3(x_2 \vee x_4x_5), f'_{x_1} = x_2\bar{x}_3x_4x_5 \vee x_2\bar{x}_3x_4\bar{x}_5 \vee \bar{x}_2\bar{x}_3x_4x_5.$$

The situation is similar to case (f).

(i) Let $f = \bar{x}_1x_2 g(x_3, \dots, x_6) \vee \; x_1 g(x_2x_3, x_4, x_5, x_6)$. Then

$$f^0_{x_6} = x_2x_3x_4 \vee x_2x_3x_5, f^1_{x_6} = g(x_2, x_3x_4, x_5, x_1).$$

Hence it follows that $\delta(f) \subsetneq \delta(f^0_{x_6})$, $\delta(f) = \delta(f^1_{x_6})$, $\delta(f^0_{x_6}) \setminus \delta(f) = \{x_1\}$ and $\delta(f) = \delta(f^0_{x_1}) = \delta(f^1_{x_1})$, that it is not true.

Thus, $S_{B_4} \cap P = \emptyset$ and $B_4 \subseteq P$. Theorem is proved.

$\square$

References

[1] Vinokurov S. F. and Peryazev N. A., *Selected Problems of the Theory of Boolean functions.* Nauka, Moscow, 2001 (in Russian).

[2] Stetsenko V. A., *On the premaximal bases in* P_2. Mat. Voprosy Kibernet., No. 4, 139–177 (1992).

[3] Subbotovskaya B. A., *Comparison of bases in the realization by formulas of functions of the algebra of logic*, Soviet Math. Dokl., No. 4, (1963).

[4] Gurvich V. A., *Repetition-free criteria of functions in the algebra of logic*, Soviet Math. Dokl., 43, No. 3, 721–726 (1991).

[5] Kirichenko K. D., *Repetition-free criteria for Boolean functions in various bases*, in: Optimization, Control, and Intellect [in Russian], Irkutsk, 2000, No. 4, pp. 93–101.

[6] Peryazev N. A., *Realization of Boolean functions by the repetition-free formulas in some bases, in: Algebra, Logic, and Applications* [in Russian], Irkutsk, 1994, pp. 143–154.

[7] Peryazev N. A., *On realization of Boolean functions by formulae without repetitions*, Discrete Math., 7, No. 3, 61–68 (1995).

[8] Peryazev N. A. and Sharankhaev I. K., *Repetition-free criteria for a Boolean function in the pre-elementary bases of rank 3*, Discrete Math., 17, No. 2, 127–138 (2005).

[9] Sharankhaev I. K., *On realization of Boolean functions by the repetition-free formulas in particular base*, Siberian Math. Journal., Vol. 50, No. 1, 231–237 (2009).

[10] Sharankhaev I. K., *On repetition-free Boolean functions in almost elementary monotone bases*, Discrete Math., 21, No. 2, 88–93 (2009).

[11] Sharankhaev I. K., *On representation of Boolean functions by the repetition-free formulas in some symmetric bases*, Vestnik of Buryat State University. Math. and Computer Science, Physics and Technics, 6, 53-56 (2007).

[12] Sharankhaev I. K., *On Boolean bases of second level*, Izvestia vuzov. Math., No. 3, 81–82 (2004).

I.K. Sharankhaev Buryat State University, Smolin street, 24a, Ulan-Ude, Russia, 670000

E-mail address: goran5@mail.ru